Preference Function Modeling (PFM): The Mathematical Foundations of Decision Theory

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Abstract

We establish the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable. The mathematical foundations of decision theory as well as other theories depend on these conditions which have not been correctly identified in the classical literature.

Keywords

Foundations of science, measurement theory, decision theory, social choice, group decision making, utility theory, game theory, economic theory, mathematical psychology.

1 Introduction

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied to enable the application of the mathematical operations of linear algebra and calculus. We identify these conditions and show that classical measurement and evaluation theories, including utility theory, cannot serve as the mathematical foundation of decision theory, game theory, economics, or other scientific disciplines since they do not satisfy these conditions. In addition, the mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables,* i.e, to variables that describe psychological or subjective properties such as *preference.*

Whether psychological properties can be measured (and hence whether mathematical operations can be applied to psychological variables) was debated by a Committee that was appointed in 1932 by the British Association for the Advancement of

Science but the opposing views in this debate were not reconciled in the Committee's 1940 Final Report.

In 1944, game theory was proposed as the proper instrument with which to develop a theory of economic behavior where utility theory was to be the means for measuring preference. We show that the interpretation of utility theory's lottery operation which is used to construct utility scales leads to an intrinsic contradiction and that the operations of addition and multiplication are not applicable on utility scale values. We present additional shortcomings of utility theory which render it unsuitable to serve as mathematical foundations for economics or other theories and we reconstruct these foundations.

2 Measurement of Preference – Empirical Addition

The applicability of mathematical operations is among the issues implicitly addressed by von Neumann and Morgenstern in [35, §§3.2–3.6] in the context of measurement of individual preferences. Preference, or value, or utility, is not a physical property of the objects being valued, that is, preference is a subjective, i.e. psychological, property. Whether psychological properties can be measured was an open question in 1940 when a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (Ferguson *et al.* [22]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the Final Report.

The position that psychological variables cannot be measured was supported by Campbell's view on the role of measurement in physics [17, Part II] which elaborated upon Helmholtz's earlier work on the mathematical modelling of physical measurement [23]. The main elements of this view are summarized by J. Guild in Ferguson *et al.* [22] in the context of measurement of *sensation*.

To re-state Campbell's position in current terminology the following is needed. By an empirical system E we mean a set of empirical *objects* together with *operations* (i.e. functions) and possibly the relation of *order* which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the empirical operations in E as well as the order in E when E is ordered. A scale s is a mapping of the objects in E into the objects in M that reflects the structure of E into M (in technical terms, a scale is a homomorphism from E into M).

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: As Campbell eloquently states [17, pp. 267–268], "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science."

In terms of these concepts, Guild [22, p. 345] states that for psychological variables it is not possible to construct a scale that reflects the empirical operation of addition because such an empirical (or "practical") addition operation has not been defined; if the empirical operation does not exist, its mathematical reflection does not exist as well.

The framework of mathematical modelling is essential. To enable the application of mathematical operations in a given empirical system, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects (which typically are the real numbers for the reasons given in §4.3). Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Therefore, the construction of measurement scales requires that the property-specific empirical operations be identified and reflected in the mathematical model. Moreover, the operations should be chosen so as to achieve the goal of this construction which is the application of mathematical operations in the mathematical model.

2.1 Empirical Addition – Circumventing the Issue

Accordingly, von Neumann and Morgenstern had to identify the empirical operations that characterize the property of *preference* and construct a corresponding mathematical model. As we shall see in §3, their empirical operation requires an interpretation that leads to an intrinsic contradiction while the operations of addition and multiplication are not enabled in their mathematical model.

The task of constructing a model for *preference* measurement is addressed by von Neumann and Morgenstern in [35, §3.4] indirectly in the context of measurement of *individual preference*. While the operation of addition as applies to *length* and *mass* results in scales that are unique up to a positive multiplicative constant, physical variables such as *time* and *potential energy* to which standard mathematical operations do apply are unique up to an additive constant and a positive multiplicative constant. (If s and t are two scales then for *time* or *potential energy* $t = p + q \times s$ for some real numbers p and $q > 0$ while for *length* or *mass* $t = q \times s$ for some $q > 0$.) This observation implies that Guild's argument against the possibility of measurement of psychological variables is not entirely correct. It also seems to indicate the need to identify an empirical – "natural" in von Neumann and Morgenstern's terminology – operation for *preference* measurement for which the resulting scales are unique up to an additive constant and a positive multiplicative constant. Seeking an empirical operation that mimics the "center of gravity" operation, they identified the now-familiar utility theory's operation of constructing lotteries on "prizes" to serve this purpose.

Von Neumann and Morgenstern's *uniqueness* argument and *center of gravity* operation are the central elements of their utility theory which is formalized in the axioms of [35, §3.6]. This theory is the basis of game theory which, in turn, was to serve as the mathematical foundation of economic theory. Elaborating upon von Neumann and Morgenstern's concepts, Stevens [39] proposed a uniqueness-based classification of "scale type" and the focus on the issues of the possibility of measurement of psychological variables and the applicability of mathematical operations to scale values has moved to the construction of "interval" scales, i.e. scales that are unique up to an additive constant and a positive multiplicative constant.

3 Shortcomings of Utility and Game Theory

The argument against the possibility of measurement of psychological variables can be rejected on the basis of the uniqueness argument but constructing utility scales that are immune from this argument is not equivalent to establishing that psychological variables can be measured. In fact, as we now show, the operations of addition and multiplication do not apply to utility scale values.

This and additional shortcomings of utility theory render it unsuitable to serve as the foundation for the application of mathematical methods in economic theory. As for game theory, we will replace its utility foundations with proper ones but additional corrections are required if it is to be the proper instrument with which to develop a theory of economic behavior (see also Barzilai [3–5, 7, 10, 12]).

3.1 Applicability of Operations on Scale Values vs. Scale Operations

Consider the applicability of the operations of addition and multiplication to scale values for a fixed scale, that is, operations that express facts such as "the weight of an object equals the sum of the weights of two other ones" (which corresponds to addition: $s(a) = s(b) + s(c)$) and "the weight of a given object is two and a half times the weight of another" (which corresponds to multiplication: $s(a) = 2.5 \times s(b)$).

It is important to emphasize the distinction between the application of the operations of addition and multiplication to scale values for a fixed scale (for example $s(a) = s(b) + s(c)$) as opposed to what appear to be the same operations when they are applied to an entire scale whereby an equivalent scale is produced (for example $t = p + q \times s$ where s and t are two scales and p, q are numbers). In the case of scale values for a fixed scale, the operations of addition and multiplication are applied to elements of the mathematical system M and the result is another element of M. In the case of operations on entire scales, addition or multiplication by a number are applied to an element of the set $S = \{s, t, \ldots\}$ of all possible scales and the result is another element of S rather than M. These are different operations because operations are functions and functions with different domains or ranges are different.

In the case of "interval" scales where the uniqueness of the set of all possible scales is characterized by scale transformations of the form $t = p + q \times s$, it cannot be concluded that the operations of addition and multiplication are applicable to scale values for a fixed scale such as $s(a) = s(b) + s(c)$. It might be claimed that the characterization of scale uniqueness by $t = p + q \times s$ implies the applicability of addition and multiplication to scale values for fixed scales, but this claim requires proof. (There is no such proof, nor such claim, in the literature because a simple argument[1] shows that this claim is false.)

3.2 The Principle of Reflection and the Utility Operation

3.2.1 The Principle of Reflection

Consider the measurement of *length* and suppose that we can only carry out ordinal measurement on a set of objects, that is, for any pair of objects we can determine which one is longer or whether they are equal in length (in which case we can order the objects by their length). This may be due to a deficiency with the state of technology (appropriate tools are not available) or with the state of science (the state of knowledge and understanding of the empirical or mathematical system is insufficient). We can still construct scales (functions) that map the empirical objects into the real numbers but although the real numbers admit many operations and relations, the only relation on ordinal scale values that is relevant to the property under measurement is the relation of order. Specifically, the operations of addition and multiplication can be carried out on the range of such scales since the range is a subset of the real numbers, but such operations are extraneous because they do not reflect corresponding empirical operations. Extraneous operations may not be carried out on scale values – they are irrelevant and inapplicable; their application to scale values is a modelling error.

The Principle of Reflection is an essential element of modelling that states that operations within the mathematical system are applicable *if and only if* they reflect corresponding operations within the empirical system. In technical terms, in order for the mathematical system to be a valid model of the empirical one, the mathematical system must be homomorphic to the empirical system (a homomorphism is a structure-preserving mapping). A mathematical operation is a valid element of the model only if it is the homomorphic image of an empirical operation. Other operations are not applicable on scale values.

By the Principle of Reflection, a necessary condition for the applicability of an operation on scale values is the existence of a corresponding empirical operation (the homomorphic pre-image of the mathematical operation). That is, the Principle of Reflection applies in both directions and a given operation is applicable in the mathematical image only if the empirical system is equipped with a corresponding operation.

3.2.2 Addition and Multiplication Are Not Applicable to Utility Scales

The Principle of Reflection implies that the operations of addition and multiplication are not applicable to utility scales despite their "interval" type. These operations are not applicable to von Neumann and Morgenstern's utility model because their axioms include *one* compound empirical *ternary* operation (i.e. the "center of gravity" operation which is a function of *three* variables) instead of the *two binary* operations of addition and multiplication (each of which is a function of *two* variables). Addition and multiplication are not enabled on utility scale values in later formulations as well because none of these formulations is based on two empirical operations that correspond to addition and multiplication. It should be noted that the goal of constructing the utility frame-

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1 Consider the automorphisms of the group of integers under addition. The group is a model of itself ($E = M$), and scale transformations are multiplicative: $t = (\pm 1) \times s$. However, by definition, the operation of multiplication which is defined on the set of scales is not defined on the group M.

work was to enable the application of mathematical operations rather than to build a system with a certain type of uniqueness.

3.3 Utility Theory's Intrinsic Inconsistency

As an abstract mathematical system, von Neumann and Morgenstern's utility axioms are consistent. However, while von Neumann and Morgenstern establish the *existence* and *uniqueness* of scales that satisfy these axioms, they do not address utility scale *construction.* This construction requires a specific interpretation of the empirical operation in the context of preference measurement and although the axioms are consistent in the abstract, the *interpretation* of the empirical utility operation creates an intrinsic inconsistency: The interpretation of the utility operation in terms of lotteries constrains the values of utility scales for lotteries while the values of utility scales for prizes are unconstrained; the theory permits lotteries that are prizes and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained. In other words, utility theory has one rule for assigning values to prizes and a different, conflicting, rule for assigning values to lotteries. For a prize which is a lottery ticket, the conflicting rules are contradictory

3.4 Shortcomings of Game Theory

Game theory's characteristic function is ill-defined; the theory employs sums that are undefined; the reduction of an n-player game to a 2-player game depends on the unsupported notion of *the utility of a group of players*; "the value" of two-person zero-sum game theory is not unique and consequently is ill-defined; and there are additional errors at the foundations of game theory that need to be corrected. For details see Barzilai [3 and 5].

4 Reconstructing the Foundations

4.1 Proper Scales – Straight Lines

In order to enable the "powerful weapon of mathematical analysis" to be applied to any scientific discipline it is necessary, at a minimum, to construct models that enable the operations of addition and multiplication for without these operations the tools of linear algebra and elementary statistics cannot be applied. This construction, which leads to the well-known geometrical model of points on a straight line, is based on two observations:

1. If the operations of addition and multiplication are to be enabled in the mathematical system M, these operations must be defined in M. The empirical system E must then be equipped with corresponding operations in order for M to be a model of E.

2. Mathematical systems with an absolute *zero* or *one* are not homogeneous: these special, distinguishable, elements are unlike others. On the other hand, since the existence of an absolute *zero* or *one* for empirical systems that characterize subjective properties has not been established, they must be modelled by homogeneous mathematical systems.

Sets that are equipped with the operations of addition and multiplication, including the inverse operations of subtraction and division, are studied in abstract algebra and are called *fields*. The axioms that define fields are listed in *Appendix A*. A field is a not a homogeneous system since it contains two special elements, namely an absolute *zero* and an absolute *one* which are the additive and multiplicative identities of the field (in technical terms, they are invariant under field automorphisms). It follows that to model a subjective property by a mathematical system where the operations of addition and multiplication are defined we need to modify a field in order to homogenize its special elements, i.e., we need to construct a *homogeneous field*. To homogenize the multiplicative identity, we construct a one-dimensional vector space which is a *partially homogeneous field* (it is homogeneous with respect to the multiplicative identity but not with respect to the additive identity) where the elements of the field serve as the set of scalars in the vector space. To homogenize the additive identity as well, we combine points with the vectors and scalars and construct a one-dimensional affine space, which is a homogeneous field, over the previously constructed vector space. The axioms characterizing vector and affine spaces are listed in *Appendix A*. The end result of this construction, the one-dimensional affine space, is the algebraic formulation of the familiar straight line of elementary (affine) geometry so that for the operations of addition and multiplication to be enabled on models that characterize subjective properties, the empirical objects must correspond to points on a straight line of an affine geometry. For details see *Appendix A,* or the equivalent formulations in Artzy [2, p. 79], and Postnikov [36, pp. 46–47].

In an affine space, the difference of two points is a vector and no other operations are defined on points. In particular, it is important to note that the ratio of two points as well as the sum of two points are undefined. The operation of addition is defined on *point differences,* which are vectors, and this operation satisfies the *group* axioms listed in *Appendix A*. Multiplication of a vector by a scalar is defined and the result is a vector. In the one-dimensional case, and only in this case, the ratio of a vector divided by another non-zero vector is a scalar.

It follows that Campbell's argument is correct with respect to the application of *The Principle of Reflection* and the identification of addition as a fundamental operation, but that argument does not take into account the role of the multiplication operation and the modified forms of addition and multiplication when the models correctly account for the degree of homogeneity of the relevant systems. Note also that it is not sufficient to model the operation of addition since, except for the natural numbers, multiplication is not repeated addition: In general, and in particular for the real numbers, multiplication is not defined as repeated addition but through field axioms.

Since the purpose of modelling is to enable the application of mathematical operations, we classify scales by the type of mathematical operations that they enable. We use the terms *proper scales* to denote scales where the operations of addition and multiplication are enabled on scale values, and *weak scales* to denote scales where these operations are not enabled. This partition is of fundamental importance and we note that it follows from *The Principle of Reflection* that all the models in the literature are weak because they are based on operations that do not correspond to addition and multiplication.

4.2 Implications: Undefined Ratios and Pairwise Comparisons

In order for the operations of addition and multiplication to be applicable, the mathematical system M must be (i) a field if it is a model of a system with an absolute *zero* and *one*, (ii) a one-dimensional vector space when the empirical system has an absolute *zero* but not an absolute *one*, or (iii) a one-dimensional affine space which is the case for all non-physical properties with neither an absolute *zero* nor absolute *one*. This implies that for proper scales, scale ratios are undefined for subjective variables including *preference*. In particular, this invalidates all decision methodologies that apply the operations of addition and multiplication to scale values and are based on preference ratios. For example, in the absence of an absolute zero for *time,* it must be modelled as a homogeneous variable and the ratio of two times (the expression t_1/t_2), is undefined. For the same reason, the ratio of two potential energies e_1/e_2 is undefined while *the ratios of the differences* $\Delta t_1/\Delta t_2$ and $\Delta e_1/\Delta e_2$ are properly defined. We saw that the sum of von Neumann and Morgenstern's utility scale values is undefined. Since the sum of two points in an affine space is undefined, the sum of proper preference scale values is undefined as well.

The expression $(a-b)/(c-d) = k$ where a, b, c, d are points on an affine straight line and k is a scalar is used in the construction of proper scales. The number of points in the left hand side of this expression can be reduced from four to three (e.g. if $b = d$) but it cannot be reduced to two and this implies that pairwise comparisons cannot be used to construct preference scales where the operations of addition and multiplication are enabled.

4.3 Strong Scales – the Real Numbers

Proper scales enable the application of the operations of linear algebra but are not necessarily equipped with the relation of order which is needed to indicate a direction on the straight line (for example, to indicate that an object is more preferable, or heavier, or more beautiful than another). To construct proper ordered scales the underlying field must be ordered (for example, the field of complex numbers is unordered while the field of the rational numbers is ordered). For a formal definition of an ordered field see McShane and Botts [31, Ch. 1, §3].

Physics, as well as other sciences, cannot be developed without the mathematical "weapons" of calculus. For example, the basic concept of acceleration in Newton's Second Law is defined as a (second) derivative; in statistics, the standard deviation requires the use of the square root function whose definition requires the limit operation; and marginal rates of change, defined by partial derivatives, are used in economics. If calculus is to be enabled on ordered proper scales, the underlying field must be an ordered field where any limit of elements of the field is itself an element of the field.

In technical terms, the underlying field must be *complete* (see McShane and Botts [31, Ch. 1, §5] for a formal definition). Since the only ordered complete field is the field of real numbers, in order to enable the operations of addition and multiplication, the relation of order, and the application of calculus on subjective scales, the objects must be mapped into the real, ordered, homogeneous field, i.e. a one-dimensional, real, ordered, affine space, and the set of objects must be a subset of points on an empirical ordered real straight line. We use the term *strong models* to denote such models and *strong scales* to denote scales produced by strong models.

The application of the powerful weapon of mathematical analysis requires a system in which addition and multiplication, order, and limits are enabled. The reason for the central role played by the real numbers in science is that the field of real numbers is the only ordered complete field.

5 Measurement Theory

Beginning with Stevens [39] in 1946, measurement theory (which only deals with the *mathematics* of measurement) has centred on issues of scale uniqueness rather than applicability of operations. As a result of the shift of focus from applicability of operations to uniqueness, the operations of addition and multiplication are not applicable on scale values for any scale constructed on the basis of this theory regardless of their "scale type" including "ratio" scales and "interval" scales as shown in §3.1 (see also Barzilai [9]).

The focus of this theory was further narrowed when Scott and Suppes [38] in 1958 adopted a system with a single set of objects as the foundation of the theory. Vector and affine spaces cannot be modelled by such systems because the construction of vector and affine spaces requires two or three sets respectively (the sets of scalars, vectors, and points). The operations on points, vectors and scalars are not closed operations: the difference of two points in an affine space is a vector rather than a point and, in a one-dimensional space, the ratio of two vectors is a scalar rather than a vector. Because proper scales for psychological variables are affine scales that are based on three sets, the operations of addition and multiplication are not enabled on scales constructed on the basis of classical measurement theory for any psychological variable for in this theory no model is based on three sets. In particular, this is the case for *preference* which is the fundamental variable of decision theory. In consequence, the mathematical foundations of decision theory must be replaced in order to enable the application of mathematical operations including addition and multiplication.

The mathematical models in *Foundations of Measurement* (Krantz *et al.* [28] and Luce *et al.* [30]) and Roberts [37] are incorrect even for the most elementary variable of physics – *position* of points on an affine straight line. Derived from the model of *position*, the correct model for *length* of segments (position differences) on this line is a one-dimensional vector space. Likewise, "extensive measurement" (see e.g. Roberts [37, §3.2]) is not the correct model for the measurement of *mass*, another elementary physical variable. In essence, "extensive measurement" is the "vector half" of a one-dimensional vector space where multiplication and the scalars are lost. Not surprisingly, the second half of a one-dimensional affine space is then lost in the classical theory's "difference measurement" where the scalars and vectors are both lost together with vector addition and scalar multiplication (see Roberts [37, §3.2–3.3]). In his 1992 paper [29, p. 80], Luce acknowledges the inadequacy of the models of the classical theory: "Everybody involved in this research has been aware all along that the class of homogeneous structures fails to include a number of scientifically important examples of classical physical measurement and, quite possibly, some that are relevant to the behavioral sciences." But despite the evidence of inadequacy, these models have not been corrected in the classical theory.

In summary, the fundamental problem of applicability of mathematical operations to scale values has received no attention in the classical theory of measurement after 1944; the theory does not provide the tools and insight necessary for identifying shortcomings and errors of evaluation and decision methodologies including utility theory and the Analytic Hierarchy Process; the basic model of Scott and Suppes [38] is flawed; and the operations of addition and multiplication are not applicable to scale values produced by any measurement theory model.

6 Classical Decision Theory

6.1 Utility Theory

Barzilai's paradox (see [§3.3 above], [10, §6.4.2] and [12, §4.2]) and the inapplicability of addition and multiplication on utility scale values imply that utility theory cannot serve as a foundation for any scientific discipline. In addition, von Neumann and Morgenstern's utility theory was not developed as, and is not, a prescriptive theory neither is it a normative theory (see [10, §6.4.3]). Moreover, the interpretation by von Neumann and Morgenstern of utility equality as a true identity precludes the possibility of indifference between a prize and a lottery which is utilized in the construction of utility scales while under the interpretation of utility equality as indifference the construction of lotteries is not single-valued and is therefore not an operation (see [10, §6.4.4]).

In the context of decision theory, despite the evidence to the contrary (e.g. Barzilai [10, §6.4.3] and [12]), utility theory is still treated by some as the foundation of decision theory and is considered a normative theory. Howard in particular refers to utility theory in the non-scientific term "The Old Time Religion" [24] while elsewhere he refers to "Heathens, Heretics, and Cults: The Religious Spectrum of Decision Aiding" [25]. A recent publication entitled "Advances in Decision Analysis" [21] does not contribute to correcting these errors.

6.2 The Analytic Hierarchy Process

The Analytic Hierarchy Process (AHP) is not a valid methodology. More than thirty years after the publication of Miller's work in the 1960s [32–34], there is still no acknowledgement in the AHP literature (or elsewhere) of his contribution to decision theory in general and the AHP in particular. Miller was not a mathematician and his methodology is based on mathematical errors although some of its non-mathematical elements are valuable. The AHP is based on these mathematical errors and additional ones (see Barzilai [8, 13–16] and the references there).

Not surprisingly, these errors have been mis-identified in the literature and some of these errors appear in decision theory. For example, Kirkwood [27, p. 53] relies on Dyer and Sarin which repeats the common error that the coefficients of a linear value function correspond to relative importance [19, p. 820]. Furthermore, "difference measurement" which is the topic of Dyer and Sarin is not the correct model of preference measurement. More specifically, in his *Remarks on the Analytic Hierarchy Process* [18, p. 250] Dyer's major focus is in §3 where he argues that the AHP "generates rank orderings that are not meaningful" and states that "[a] symptom of this deficiency is the phenomenon of rank reversal" but his argument is circular since the only AHP deficiency presented in §3 of his paper is rank reversal. Moreover, the AHP suffers from multiple methodological flaws that cannot be corrected by "its synthesis with the concepts of multiattribute utility theory" (which suffers from its own flaws) as stated by Dyer [18, p. 249].

The AHP is a method for constructing preference scales and, as is the case for other methodologies, the operations of addition and multiplication are not applicable on AHP scale values (see §3.1 above or Barzilai [9]). The applicability of addition and multiplication must be established *before* these operations are used to compute AHP eigenvectors and, as we saw in §3.1, the fact that eigenvectors are unique up to a multiplicative constant does not imply the applicability of addition and multiplication.

In order for addition and multiplication to be applicable on preference scale values, the alternatives must correspond to points on a straight line in an affine geometry (see §4.1 or Barzilai [11]). Since the ratio of points on an affine straight line is undefined, preference ratios, which are the building blocks of AHP scales, are undefined. In addition, pairwise comparisons cannot be used to construct affine straight lines.

The fundamental mathematical error of using inapplicable operations to construct AHP scales renders the numbers generated by the AHP meaningless. Other AHP errors include the fact that the coefficients of a linear preference function cannot correspond to weights representing relative importance and therefore cannot be decomposed using Miller's criteria tree; the eigenvector method is not the correct method for constructing preference scales; the assignment of the numbers 1-9 to AHP's "verbal scales" is arbitrary, and there is no foundation for these "verbal scales" (see Barzilai [6, 8, 13–16]).

6.3 Value Theory

Scale construction for physical variables requires the specification of the property under measurement and the empirical objects. For example, if the property under measurement is *temperature*, the construction results in a *temperature* scale and, clearly, the measurement of *length* does not produce a *mass* scale. In the case of subjective measurement too, the property under measurement must be explicitly specified. If the property under measurement is *preference*, the resulting scales are *preference* scales. It follows that *preference, utility, value,* and *priority* are synonyms for the same underlying subjective property and the distinction between utility theory and value theory has no scientific basis. It also follows that Keeney and Raiffa's notion of "the utility of value" ($u[v(x)]$, see [26, p. 221])is as meaningless as "the temperature of temperature" or "the length of length" are.

Likewise, although the notions of "strength of preference" (Dyer and Sarin [19]) and "difference measurement" (e.g. Krantz *et al.* [28], Roberts [37]) are intuitively appealing, these measurement models of *value, utility, priorities,* etc., are based on measurement theory errors as shown above. Similarly, the utility theories in Edwards [20] are founded on errors as well and, although the issues have been known for a few years, the more recent "Advances in Decision Analysis" (Edwards *et al.* [21]) does not contribute to correcting these methodological errors.

6.4 Group Decision Making

The common view in the classical literature that group decision making cannot be modelled mathematically is an error that is based on a misinterpretation of the implications of Arrow's Impossibility Theorem [1]. Arrow's impossibility theorem is a negative result and the construction of preference scales cannot be founded on negative results. In addition, this theorem deals with ordinal scales which enable the relation of order but do not enable the operations of addition and multiplication. The concepts of trade-off and cancellation are not applicable to ordinal scales – see Barzilai [10, §6.5] for details.

7 Summary

Classical decision and measurement theory are founded on errors which have been propagated throughout the literature and have led to a large number of methodologies that are based on flawed mathematical foundations and produce meaningless numbers. The fundamental issue of applicability of the operations of addition and multiplication to scale values was not resolved by von Neumann and Morgenstern's utility theory and the literature of classical decision and measurement theory offers no insight into this and other fundamental problems. Decision analysis is not a prescriptive theory and will not be a sound theory until these errors are corrected.

We identified the conditions that must be satisfied in order to enable the application of linear algebra and calculus, and established that there is only one model for strong measurement of subjective variables. The mathematical foundations of the social sciences need to be corrected to account for these conditions. In particular, foundational errors in utility theory, game theory, mathematical economics, decision theory, measurement theory, and mathematical psychology need to be corrected.

This paper includes the results of very recent research. The development of the theory, methodology, and software tools continues and updates will be posted at www.scientificmetrics.com.

Appendix A: The Axioms of an Affine Straight Line

Groups and Fields

A *group* is a set G with an operation that satisfies the following requirements (i.e. axioms or assumptions):

- The operation is *closed:* the result of applying the operation to any two elements a and b in G is another element c in G. We use the notation $c = a \circ b$ and since the operation is applicable to pairs of elements of G, it is said to be a binary operation.
- The operation is *associative:* $(a \circ b) \circ c = a \circ (b \circ c)$ for any elements in G.
- The group has an *identity:* there is an element e of G such that $a \circ e = a$ for any element a in G.
- *Inverse elements:* for any element a in G, the equation $a \circ x = e$ has a unique solution x in G. If $a \circ x = e$, x is called the inverse of a.

If $a \circ b = b \circ a$ for all elements of a group, the group is called *commutative*. We reemphasize that a group is an algebraic structure with *one* operation and we also note that a group is not a homogeneous structure because it contains an element, namely its identity, which is unlike any other element of the group since the identity of a group G is the only element of the group that satisfies $a \circ e = a$ for all a in G.

A *field* is a set F with two operations that satisfy the following assumptions:

- The set F with one of the operations is a commutative group. This operation is called *addition* and the identity of the additive group is called zero (denoted '0').
- The set of all non-zero elements of F is a commutative group under the other operation on F. That operation is called *multiplication* and the multiplicative identity is called one (denoted '1').
- For any element a of the field, $a \times 0 = 0$.
- For any elements of the field the *distributive* law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

Two operations are called addition and multiplication only if they are related to one another by satisfying the requirements of a field; a single operation on a set is not termed addition nor multiplication. The additive inverse of the element a is denoted $-a$, and the multiplicative inverse of a non-zero element is denoted a^{-1} or $1/a$. Subtraction and division are defined by $a - b = a + (-b)$ and $a/b = a \times b^{-1}$.

Vector and Affine Spaces

A vector space is a pair of sets (V, F) together with associated operations as follows. The elements of F are termed *scalars* and F is a field. The elements of V are termed *vectors* and V is a commutative group under an operation termed vector addition. These sets and operations are connected by the additional requirement that for any scalars $j,\ k \in F$ and vectors $u,\ v \in V$ the scalar product $k \cdot v \in V$ is defined and satisfies, in the usual notation, $(j+k)v = jv + kv$, $k(u+v) = ku + kv$, $(jk)v = j(kv)$ and $1 \cdot v = v$.

An *affine space* (or a *point space*) is a triplet of sets $(P,\ V,\ F)$ together with associated operations as follows (see also Artzy [2] or Postnikov [36]). The pair $(V,\ F)$ is a vector space. The elements of P are termed *points* and two functions are defined on points: a one-to-one and onto function $h : P \to V$ and a "difference" function $\Delta : P^2 \to V$ that is defined by $\Delta(a,\ b) = h(a) - h(b)$. Note that this difference mapping is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \to V$, the sets of points and vectors are equipped with different operations and the operations of addition and multiplication are not defined on points. If $\Delta(a,\ b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that a point space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences*. Note that in an affine space no point is distinguishable from any other.

The dimension of the affine space $(P,\ V,\ F)$ is the dimension of the vector space V. By a homogeneous field we mean a *one-dimensional* affine space. A homogeneous field is therefore an affine space $(P,\ V,\ F)$ such that for any pair of vectors $u,\ v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$. In a homogeneous field $(P,\ V,\ F)$ the set P is termed a *straight line* and the vectors and points are said to be collinear. Division in a homogeneous field is defined as follows. For $u,\ v \in V$, $u / v = \alpha$ means that $u = \alpha v$ provided that $v \neq 0$. Therefore, in an affine space, the expression $\Delta(a,\ b) / \Delta(c,\ d)$ for the points $a,\ b,\ c,\ d \in P$ where $\Delta(c,\ d) \neq 0$, is defined and is a scalar:

$$\frac{\Delta(a,\ b)}{\Delta(c,\ d)} \in F \tag{1}$$

if and only if the space is one-dimensional, i.e. it is a straight line or a homogeneous field. When the space is a straight line, $\Delta(a,\ b)/\Delta(c,\ d) = \alpha$ (where $\Delta(c,\ d) \neq 0$) means by definition that $\Delta(a,\ b) = \alpha\Delta(c,\ d)$.

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