Game Theory Foundational Errors – Part IV

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Abstract

The minimax rule of two-person zero-sum game theory prescribes risky choices with choice probabilities that are divorced from choice consequences. The optimal probabilities cannot be computed from indefinite utility scale values and the problem is formulated incorrectly.

1 Introduction

In "Game Theory Foundational Errors – Part III" (Barzilai [1]) we showed that even if the difficulties with utility theory are ignored and the entries in a two-person zero-sum table represent outcomes rather than utility values, a small perturbation in such a game leads to a qualitative discontinuity and the minimax rule prescribes a strategy that is neither conservative nor rational. In this paper we show that the minimax strategies are not conservative in games with large probabilities as well. In addition, we note that since the linear programs that are used to compute the "optimal" probabilities have no knowledge of what the numbers in the tables represent, these numbers could be utility values. That the numbers must represent utility values rather than outcomes should be clear from Barzilai [2 and 3] and the discussion in [1, §4] (see also Remark 1.1 in Hart [4, p. 23]).

We show that when the numbers represent utility values the minimax rule prescribes choice probabilities that are divorced from choice consequences and that, in fact, the problem is formulated incorrectly and the probabilities cannot be computed from indefinite utility scale values.

2 Prescribing Risky Choices with Large Probabilities

Denote the outcomes of a two-person zero-sum game by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1}$$

where player 1 can choose between R1 and R2 (rows) and player 2 between C1 and C2 (columns) and consider the case where player 1's utility values for these outcomes are given by

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}. \tag{2}$$

According to the minimax rule, player 1 is to play R1 or R2 with probabilities (.75, .25), which are not small probabilities, while player 2's probabilities are (.5, .5). In this game if player 1 plays R1, he will end up with one of the intermediate outcomes A or B and is guaranteed at least $u(A) = 1$. If he plays R2 he will end up with either the best or the worst outcome so that R2 is the "all or nothing" choice. The minimax rule prescribes choosing the risky alternative R2 with probability 25% where R2 results in one of the extreme outcomes C or D, rather than accepting the conservative pure strategy R1 (choosing R1 with probability 100%) which leads to the intermediate outcomes A or B. As in Barzilai [1], we see that the minimax rule prescribes to player 1 a strategy that cannot be described as conservative and, for the game (2), the probability that player 1 will end up with the worst case is not small.

3 Divorcing Choice Probabilities from Choice Consequences

The optimal probabilities for the game defined by Equations (1–2) are the solutions of linear programs that are constructed from the values in (2) – no other information is used in producing these probabilities (see e.g. Hillier and Lieberman [5, Chapter 12]). Regardless of what the outcomes A, B, C, D represent, in the game (1), so long as the utility values satisfy $u(A) = 1$, $u(B) = 2$, $u(C) = 3$, and $u(D) = 0$ or, in a more convenient notation

$$u(D, A, B, C) = (0, 1, 2, 3), \tag{3}$$

player 1's probabilities are (.75, .25) and player 2's probabilities are (.5, .5).

Since the optimal probabilities depend only on the utility values in Equation (3), we now investigate the structure of outcomes that satisfy this equation. Utility values that enable the operations of the linear programs which are used to compute the optimal probabilities are unique up to an additive and a positive multiplicative constant. Adding a constant to the values in Equation (3) does not change the resulting probabilities and the same holds if the values in Equation (3) are multiplied by a positive constant. This means that the utility values

$$u(D, A, B, C) = (0, 1, 2, 3),$$

$$u(D, A, B, C) = (-5, -3, -1, 1),$$

$$u(D, A, B, C) = (871, 873, 875, 877),$$

and

$$u(D, A, B, C) = (12655, 25155, 37655, 50155),$$

are equivalent and they all yield the probabilities (.75, .25) for player 1 and (.5, .5) for player 2. Equation (3) is equivalent to

$$u(C) - u(B) = u(B) - u(A) = u(A) - u(D) \tag{4}$$

in the sense that the utility assignment that satisfies Equation (3) satisfies Equation (4), and for any utility assignment that satisfies Equation (4) there exist constants p and q such that $p + qu$ satisfies Equation (3).

In Equation (4) two of the outcomes rather than two of the numbers can be chosen arbitrarily, i.e., two of the four outcomes *A, B, C,* and *D* can be chosen arbitrarily and the other two are then constrained by Equation (4). For example, *C* and *D* may be any two possible outcomes where *C* is preferred to *D,* provided *A* and *B* are two intermediate outcomes with utility values that are equally spaced in between so that they satisfy Equation (4). This can be restated by saying that these scale assignments tell us nothing about two of the outcomes while giving us information about the other outcomes relative to these two. In all of these cases, the size of the number $u(D)$ does not indicate whether the utility of *D* is low, very low, or high.

It is necessary to emphasize that subject to the requirement that outcome *C* be preferred to *D* and the utility values of outcomes *A, B* satisfy Equation (4), player 1 is to choose R2 with probability 25% *regardless of what the outcomes C and D are!*

In other words, player 1 is to risk ending up with outcome *D* regardless of whether this outcome is attractive but less desirable than the others, or unattractive, or catastrophic. Since player 1 is to choose between R1 and R2 with probabilities (.75, .25) regardless of what the outcomes *C* and *D* are, we see that under the minimax rule, when the numbers represent utility values as they must, the "optimal" choice probabilities are divorced from choice consequences.

The choice of risk attitude should be left to the decision maker and a decision rule that divorces choice probabilities from choice consequences is not a reasonable ("rational") prescription.

4 Probabilities Must Be Assigned to Definite Scale Values

Denote by x the probability that the temperature in a given process will reach 20 degrees. Then x must depend on the choice of the temperature scale – it cannot be the case that the temperature will reach 20 degrees on the Celsius scale, the Fahrenheit scale, and any arbitrary other scale with the same probability. Probabilities cannot be assigned to scale values on an indefinite scale; when the scale is changed the probabilities assigned to scale values must change.

The reason for the divorce of choice probabilities from choice consequences in two-person zero-sum games is that probabilities are assigned to indefinite scale values. This is a fundamental error which indicates that this problem is formulated incorrectly.

5 Conclusions

In general, the "sum of outcomes" is undefined. For the sum in "zero-sum" games to be defined, its elements must be numbers. Since these numbers are added and multiplied in the linear programs that produce the "optimal" probabilities, they must represent preference scale values ("utilities") that enable the operations of addition and multiplication. A deeper argument that leads to the same conclusion is given in Barzilai [2].

When the game's "payoff" entries are defined correctly, they represent indefinite utility (i.e. preference) values, but game theory's derivation of the optimal probabilities is formulated incorrectly since these probabilities cannot be computed from indefinite utility scale values.

References

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[4] Sergiu Hart, "Games in Extensive and Strategic Form," in *Handbook of Game Theory,* Vol. 1, Robert J. Aumann and Sergiu Hart (Eds.), North-Holland, pp. 19–40, 1992.

[5] Frederick S. Hillier and Gerald J. Lieberman, *Introduction to Operations Research*, Fourth Edition, Holden-Day, 1986.